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On the Interpretation of Error Contours

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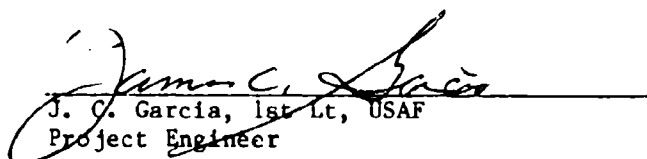DEC 16 1980
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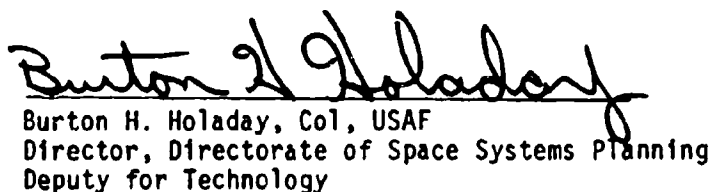
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20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

Although frequently encountered in engineering problems, the meaning of error contours has been the subject of much debate and confusion among engineers. This report attempts to clarify this situation by developing results from first principals without rigorous derivation.

While the material presented here may be well known among a miniscule clique of theorists, it appears that this exposition is required by the engineering community.

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I. INTRODUCTION

Although contours such as circular error probabilities (CEPs), error ellipses, etc., are frequently encountered in engineering problems, their meaning is often misunderstood. For example, it is not common knowledge in the engineering community that a 1σ (three-dimensional) ellipsoid carries a __different__ probability confidence level than a 1σ (two-dimensional) ellipse, or for that matter a 1σ (one-dimensional) line segment. Also, there is confusion over the relationship between elliptical and spherical error contours.

This report attempts to clarify this situation. To help explain, results are presented heuristically, without rigorous derivation. The discussion begins with a brief review of the properties of the "normal" probability distribution.

II. THE NORMAL CURVE

The most commonly encountered probability distribution in statistics is the so-called "normal" or "Gaussian" distribution presented in Fig. 1. Here, the abscissa corresponds to the observations y. The total area under the curve is equal to unity and the area lying between specified abscissa points is a measure of the relative frequency of observations between these limits.

The normal curve is completely specified by two parameters, the mean μ and the standard deviation σ. Note that the curve is "bell-shaped" and symmetric with respect to a line drawn perpendicular to the abscissa at the mean.

The standard deviation σ defines the spread of the curve. Figure 2 exhibits three normal curves having equal means but different standard deviations. Curve A with the largest spread has the largest standard deviation and Curve C the smallest. The symmetric property of the normal distribution requires that 50 percent of the observations fall below the mean and 50 percent above it. In terms of probability, the probability for an observation falling either above or below the mean is 0.50. Approximately 34 percent of the observations lie in the interval μ to $\mu + \sigma$. Thus, from the symmetric property, the probability of an observation falling between $\mu - \sigma$ and $\mu + \sigma$ is 0.68.

Since μ and σ uniquely define a normal distribution, unique probabilities can be assigned to all regions within this distribution. Thus, Table 1 presents probabilities p (k) associated with selected $\mu - k\sigma$ to $\mu + k\sigma$ limits. This table indicates, for example, that 95 percent of the observations lie in the region specified by k = 1.960 from $\mu - 1.960\ \sigma$ to $\mu + 1.960\ \sigma$.

Thus far the discussion has been limited to probabilities associated with a normally distributed scalar variable. In the next section, the scope of the discussion is extended to encompass variables of two and three dimensions.

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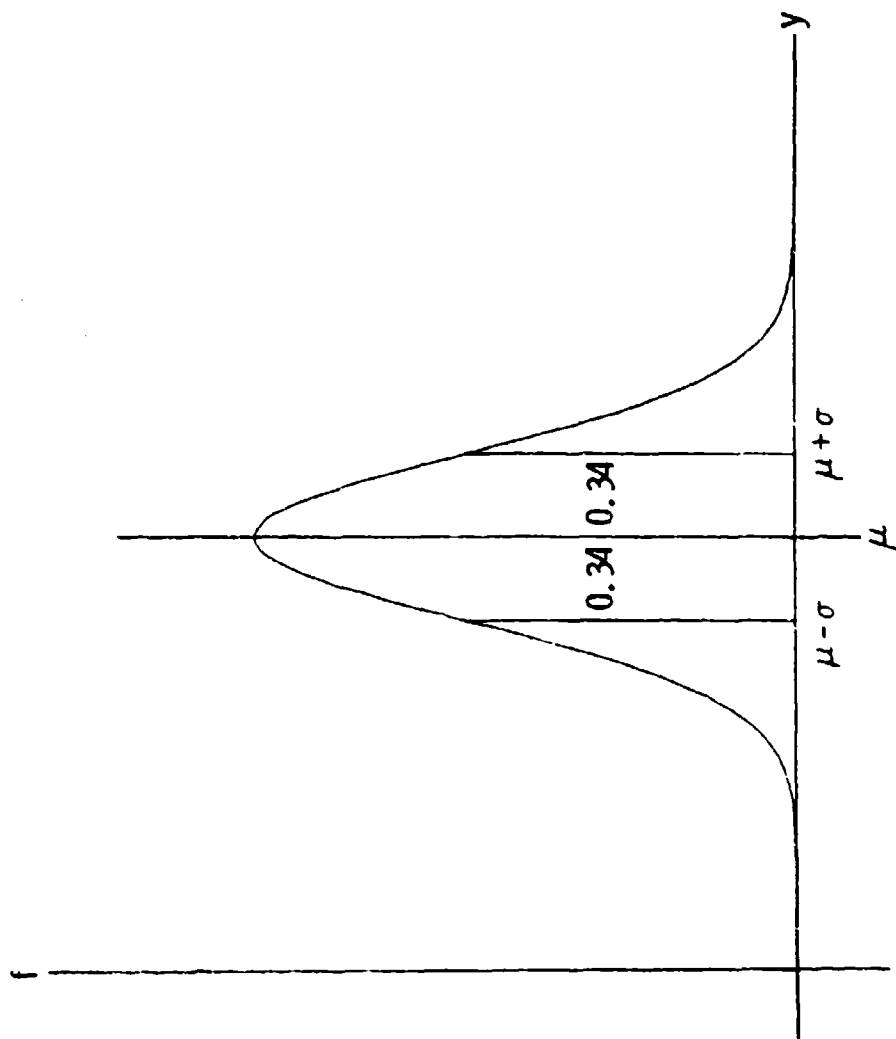
Fig. 1. Normal Distribution
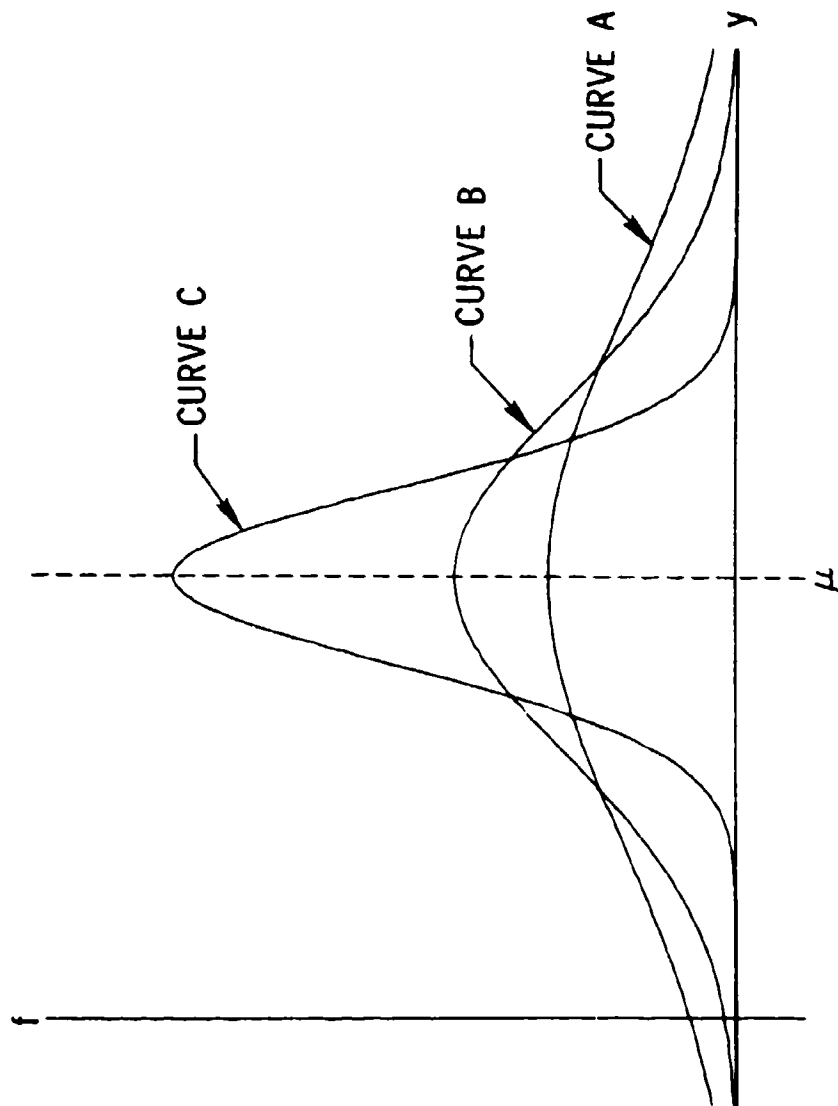
Fig. 2. Three Normal Curves with the Same Mean and Different
Standard Deviations

Table 1. Table of p(k) vs. k for One-Dimensional Normal Distribution

p(k)	k
0.05	0.0627
0.10	0.126
0.15	0.189
0.20	0.253
0.25	0.319
0.30	0.385
0.35	0.454
0.40	0.524
0.45	0.598
0.50	0.674
0.55	0.755
0.60	0.842
0.65	0.935
0.70	1.036
0.75	1.156
0.80	1.282
0.85	1.440
0.90	1.645
0.95	1.960

μ

$\mu - k\sigma$ $\mu + k\sigma$
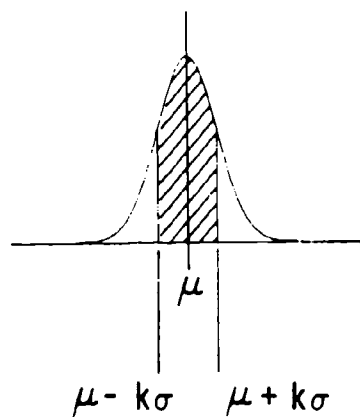
III. ERROR ELLIPSES AND ELLIPSOIDS

Let x and y denote perpendicular axes lying in a plane. Assume that independent normally distributed observations along these axes have means μ_x and μ_y and standard deviations σ_x and σ_y, respectively (Fig. 3). The 1σ error ellipse, pictured in Fig. 3 is defined as that ellipse centered at (μ_x, μ_y) with principal semiaxes of σ_x and σ_y. Similarly, for any real number k, the $k\sigma$ error ellipse (k = 1,2,3, ...) is centered at (μ_x, μ_y) with principal semiaxes of $k\sigma_x$ and $k\sigma_y$.

In an analogous manner, let x, y, and z define three orthogonal coordinates in space. Again, assume independent normally distributed observations along these axes with means denoted by μ_x, μ_y, μ_z and standard deviations by σ_x, σ_y, σ_z (Fig. 4). Then, the $k\sigma$ error ellipsoid is defined as that ellipsoid centered at (μ_x, μ_y, μ_z) with principal semiaxes $k\sigma_x$, $k\sigma_y$, $k\sigma_z$. One octant of a 1σ error ellipsoid is given in Fig. 4.

Let $p_n(k)$ denote the probability that a given observation falls within a $k\sigma$ error contour of n-dimensions; n of 1, 2, and 3 corresponding to line segments, ellipses and ellipsoids, respectively. This probability is obtained directly from the so-called "chi-square" probability distribution with n degrees-of-freedom; n = 1 yielding the normal distribution discussed in the previous section.

Tables 2 and 3, taken from Ref. 1, extend the one-dimensional results of Table 1 to two and three-dimensions. These tables reveal that probability confidence levels decrease with increasing dimension. Thus, for example, the 1σ levels are:

$$p_1 (1) = 68\% - \text{line segment (one-dimension)}$$
$$p_2 (1) = 39\% - \text{ellipse (two-dimensions)}$$
$$p_3 (1) = 20\% - \text{ellipsoid (three-dimensions)}$$

Here, specifically, $p_1 (1)$ is the probability that a <u>single</u> random variable lies within its 1σ limits. However, $p_2 (1)$ and $p_3 (1)$ are probabilities that <u>all</u> random components lie within the bounding ellipse or ellipsoid. In short, for n > 1, $p_n (1)$ assumes simultaneity, while $p_1 (1)$ does not. Indeed, $p_n (k)$ is always smaller than $p_1 (k)$ (Fig. 5).
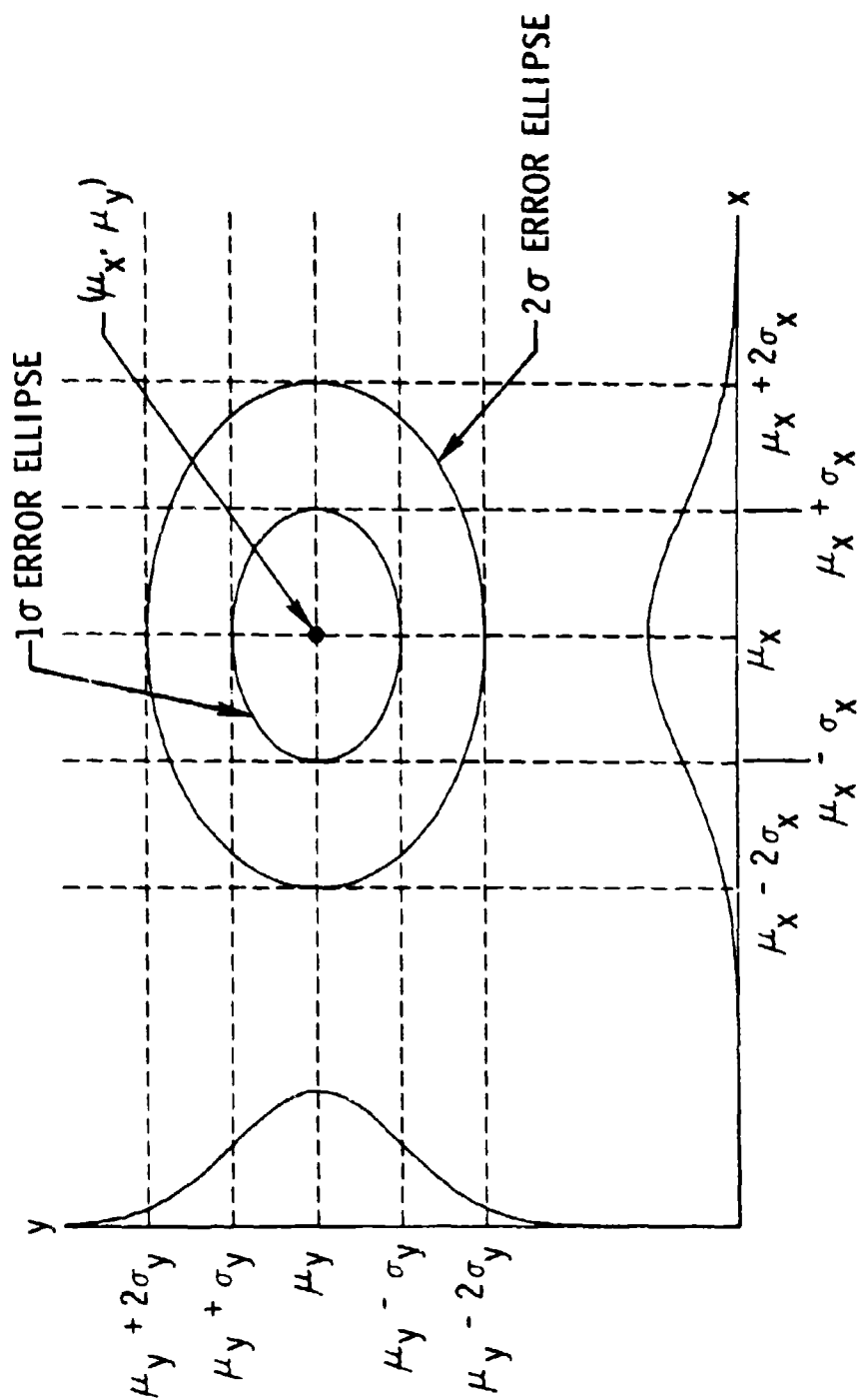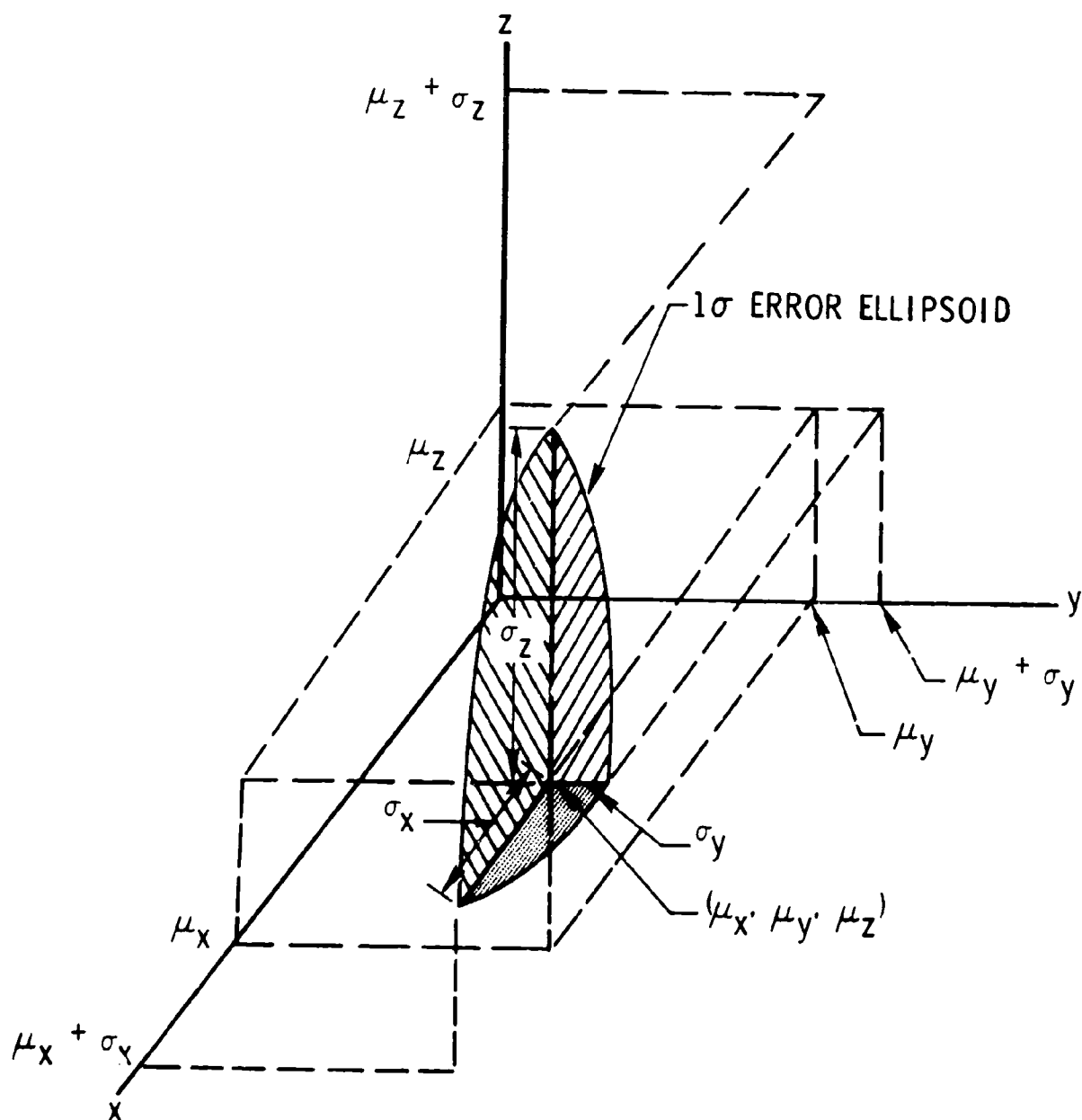
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Fig. 3. 1 and 2σ Error Ellipses

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Fig. 4. 1σ Error Ellipsoid

Table 2. Table of k for Selected $p_n(k)$

		n	
$p_n(k)$	1	2	3
0.05	0.0627	0.320	0.593
0.10	0.126	0.459	0.754
0.15	0.189	0.570	0.893
0.20	0.253	0.668	1.003
0.25	0.319	0.759	1.101
0.30	0.385	0.844	1.193
0.35	0.454	0.928	1.281
0.40	0.524	1.011	1.367
0.45	0.598	1.093	1.452
0.50	0.674	1.177	1.538
0.55	0.755	1.264	1.626
0.60	0.842	1.354	1.716
0.65	0.935	1.449	1.812
0.70	1.036	1.552	1.914
0.75	1.156	1.665	2.027
0.80	1.282	1.794	2.154
0.85	1.440	1.948	2.306
0.90	1.645	2.146	2.500
0.95	1.960	2.448	2.796
0.98	2.326	2.797	3.136
0.99	2.576	3.035	3.368
0.995	2.807	3.255	3.583

Table 3. Table of $p_n(k)$ for Selected k

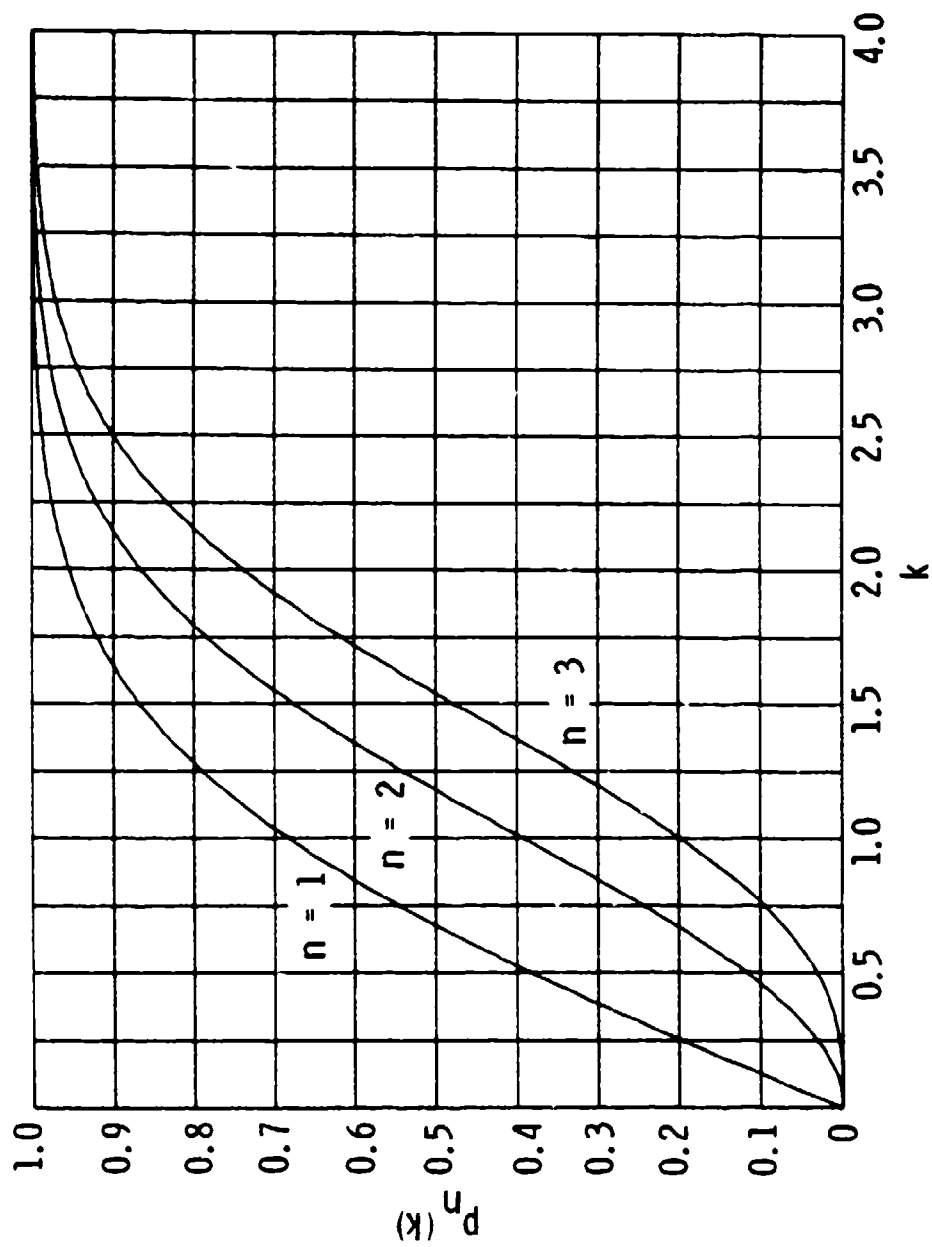|	n		
k	1	2	3
0.2	0.158	0.0198	0.00210
0.4	0.311	0.0769	0.0162
0.6	0.451	0.164	0.0516
0.8	0.576	0.274	0.113
1.0	0.683	0.393	0.199
2.0	0.954	0.865	0.739
3.0	0.997	0.989	0.971
4.0	1.000	1.000	0.999
5.0	1.000	1.000	1.000

Fig. 5. $p_n(k)$ vs. k for $n = 1, 2, 3$

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IV.   SPHERICAL ERROR CONTOURS

One disadvantage of displaying results in terms of error ellipses or ellipsoids is that several numbers are required. For example, the specification of an error ellipse requires three numbers, the magnitude of the two principal axes and an orientation angle. Similarly, six numbers are required to specify an ellipsoid, the magnitudes of the three principal axes and three orientation angles.

A possible alternative is to present results in terms of circles or spheres having a particular probability level. Although this approach may seem appealing at first, it has some disadvantages. Unless the principal axes of the ellipse or ellipsoid are nearly equal, use of the corresponding circle or sphere can lead to serious errors in engineering judgment, since all information regarding preferred directions is lost. In addition, the desired radius is not easily determined. Special algorithms for this computation are given (Refs. 2, 4). The cases where p_2 (k) = 0.50 and p_3 (k) = 0.50 correspond, respectively, to the well-known circular error probability (CEP) and spherical error probability (SEP). From Table 2, these last carry probability levels equal to those of the 1.177-σ ellipse and the 1.538-σ ellipsoid.

A procedure for computing both the 50 and 95 percent probability circles from the principal semiaxes of the 1σ ellipse is presented in Appendix A. This technique uses linear interpolation between values of the principal semiaxes and radius as listed in the extensive tables of Ref. 3.

V. COMPARISON OF TWO-DIMENSIONAL ERROR CONTOURS

The analog in two-dimensions to the one-dimensional Gaussian distribution in Fig. 1 is presented in Fig. 6. Maintaining previous notation, x and y lie along the principal axes of the error ellipse, and f is the frequency of the observations, while x´ and y´ are the observation axes in the x-y plane. It is observed in this figure, and indeed in general, that principal and observation axes are not collinear. In statistical terms, this means the observations are correlated. The probability surface is depicted as a "mountain" symmetric about the two planes containing the f-axis and one principal axis.

The total volume under the surface is equal to unity. Thus, the volume intercepted by circular or elliptical cylinders with axes parallel to f is a measure of the frequency of observations within the enclosed area on the x´ - y´ plane. Of course, circular cylinders yield circular intercepts such as the CEP and 95 percent probability circle, while error ellipses result from elliptical cylinders.

Figure 7 exhibits a family of ellipses for the two-dimensional normal distribution of Fig. 6. These ellipses, all of which have major to minor axis ratios of five to one, are obtained by slicing the probability "mountain" parallel to the x-y plane at various heights along the f-axis.

Five of the more commonly encountered contours associated with this probability distribution; the 1, 2 and 3σ ellipses, CEP, and 95 percent circle, are illustrated in Fig. 8. Superimposed upon this figure are computer-generated points randomly selected from a population with the same probability distribution.

One immediately notes that the elliptical representations contain much more information about the distribution of the points. Thus, for this example, the 3σ ellipse encloses more points than does the 95 percent circle while requiring only about half the area. As a general rule, elliptical contours are preferable except when the magnitudes of the principal axes are nearly equal.
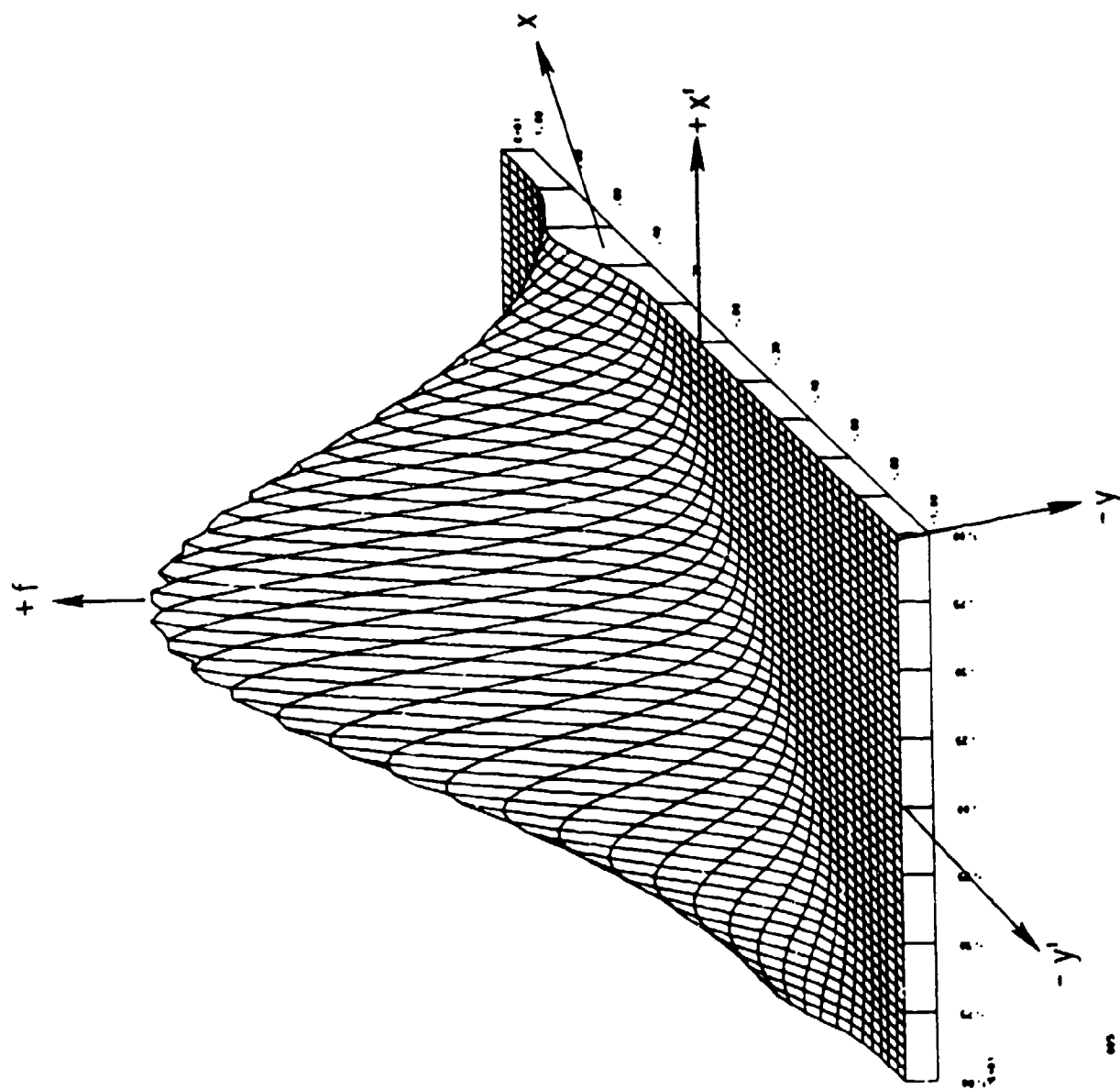
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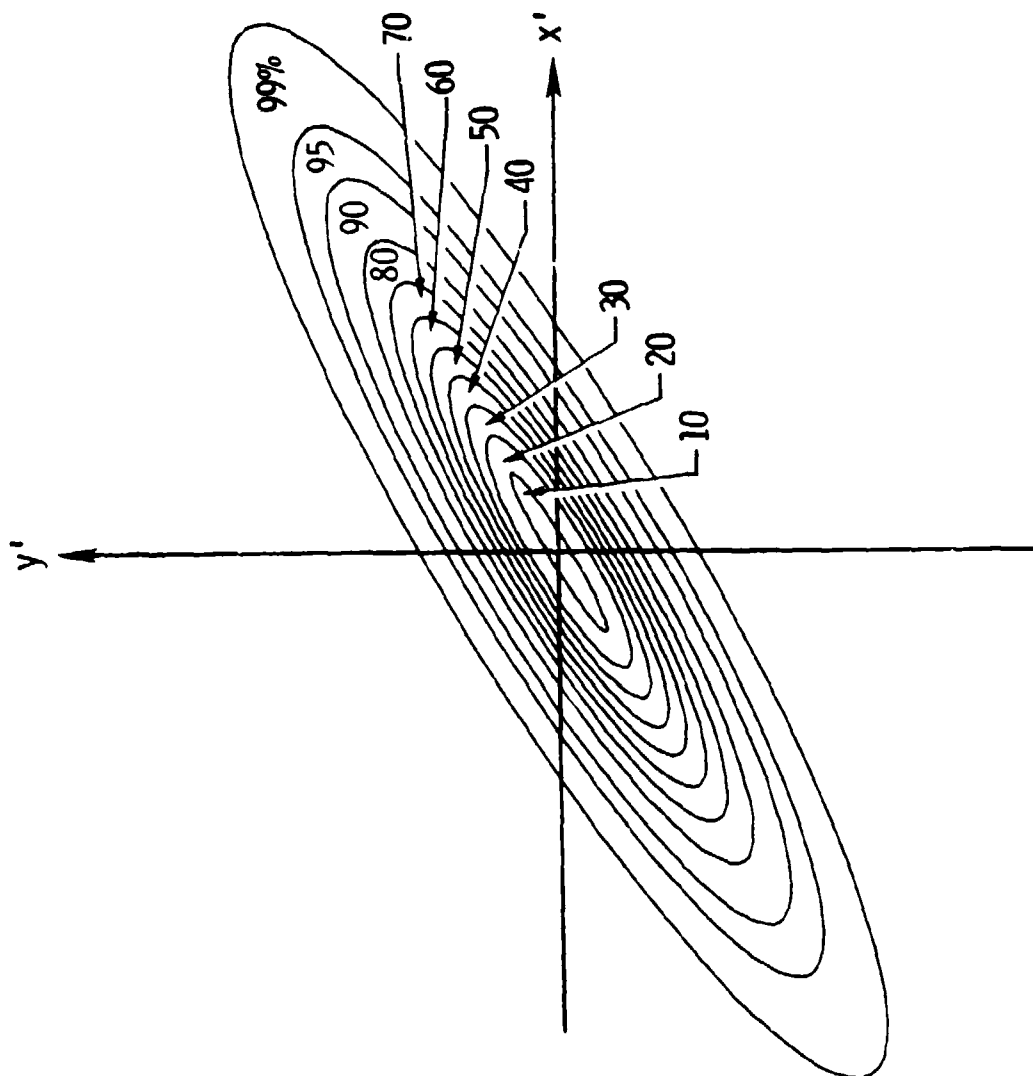Fig. 6. Two-Dimensional Gaussian Probability Distribution
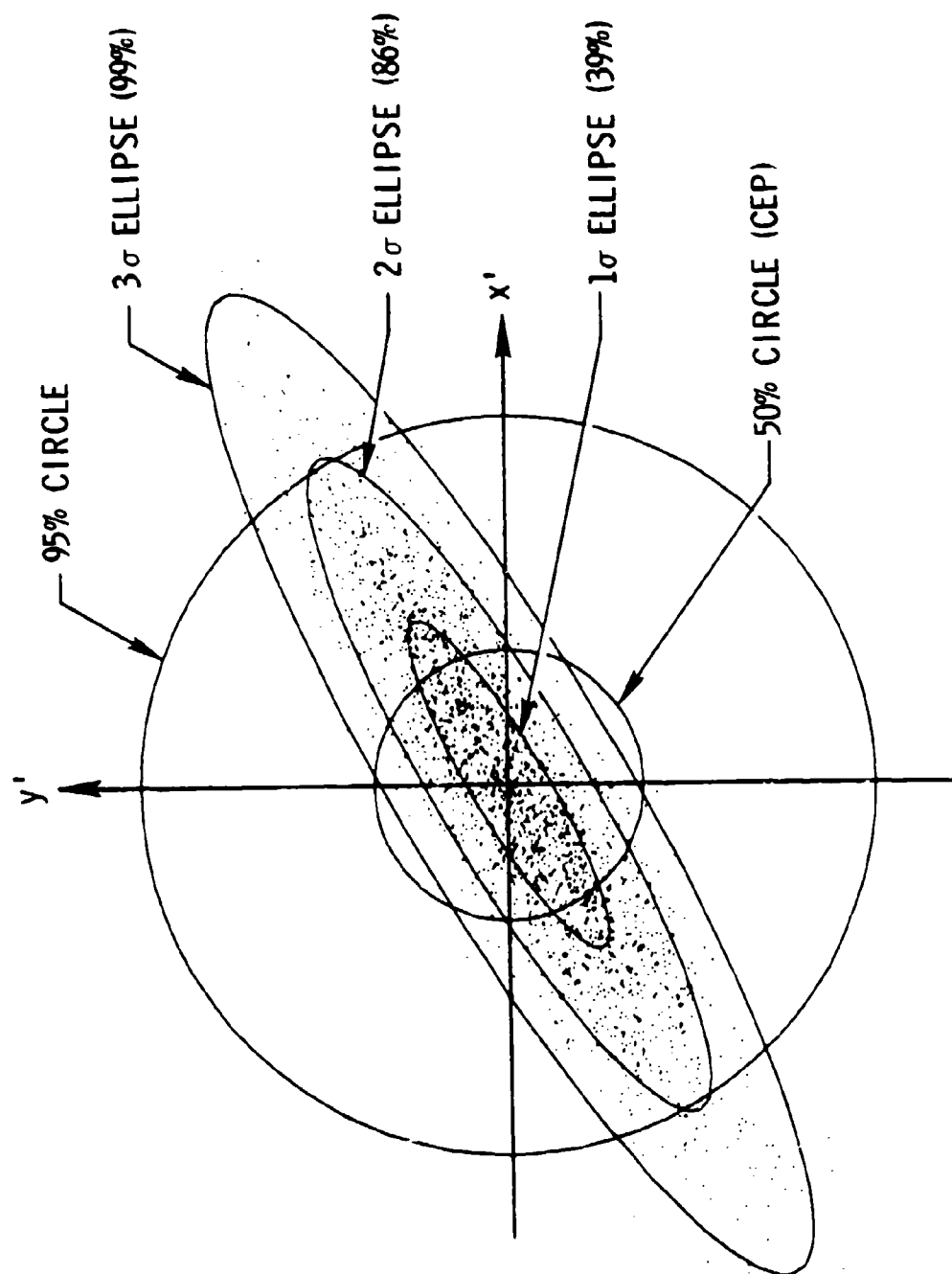
Fig. 7. Gaussian Error Ellipses

Fig. 8. Common Error Contours for Two-Dimensional Gaussian Distribution

95% CIRCLE

3σ ELLIPSE (99%)

2σ ELLIPSE (86%)

x'

1σ ELLIPSE (39%)

50% CIRCLE (CEP)

y'

VI. SUMMARY

In ascertaining the relative merits of elliptical vs. spherical error contours in reporting system errors, the more weighty arguments can be advanced in favor of the former. Not only do the error ellipse and ellipsoid convey more information than the CEP and SEP, they are actually easier to compute. The sole argument favoring the latter pair is that their specification requires only a single number.

However, the utilization of 1σ ellipses (or ellipsoids) can be very misleading to the uninitiated. Thus, the temptation to assume that the 1σ ellipse encloses at least a majority of cases, as does the one-dimensional 1σ line segment, is overwhelming. The truth is that the 1σ ellipse subtends only 39 percent of the cases. The situation is even more exaggerated in three-dimensions, where the 1σ ellipsoid encloses only 20 percent of the cases. The alternative is clear. Instead of using 1σ ellipses (or ellipsoids) one should use 50 or 95 percent ellipses (or ellipsoids). Here the name itself indicates the percentage of cases subtended and no confusion results.

APPENDIX

COMPUTATION OF 50 AND 95 PERCENT PROBABILITY CIRCLES
FROM THE PRINCIPAL AXES OF THE 1σ ERROR ELLIPSE

This appendix contains a technique for computing the radii of the 50 and
95 percent probability circles, R50 and R95, from the major and minor semiaxes
of the 1σ error ellipse, denoted λ maj and λ min respectively. In particular,
the desired radius R is obtained by linear interpolation between values of
λ maj, λ min and probability as listed in the tables of Ref. 3. The use of
linear interpolation is justified by the observation that the normalized
quantities λ min / λ maj and R/ λ maj are nearly linearly related over wide
regions.

Begin by defining the four 10-dimensional vectors

Y50 = (0.6820, 0.7059, 0.7499, 0.8079, 0.8704, 0.9336,
          0.9962, 1.0577, 1.1181, 1.1774)

SL50 = (0.0075, 0.02386, 0.04408, 0.05792, 0.06256,
          0.06323, 0.06256, 0.06149, 0.06036, 0.05935)

Y95 = (1.962, 1.970, 1.984, 2.005, 2.036, 2.081, 2.146,
          2.230, 2.332, 2.448)

SL95 = (0.006, 0.008, 0.014, 0.021, 0.031, 0.045, 0.065,
          0.084, 0.102, 0.116)

where the Y's and SL's represent values of R / λ maj and slope associated with
intervals of 0.1 in λ min / λ maj for the 50 and 95 percent radii. The proper
component of the Y and SL vectors, I, and the interpolation interval, DEL, are
computed from

X = 10 λ min / λ maj

I = Integer part of X + 1 or 10 whichever is smaller

DEL = X - I

Then the desired radii result directly from linear interpolation via

R50 = [Y50(I) + SL50(I) DEL] λ maj

R95 = [Y95(I) + SL95(I) DEL] λ maj

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